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# Second Hankel Determinent For Certain Classes Of Univalent Functions Leading To Starlike And Other Classes

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**ABSTRACT:**

Let $A_1$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disc given by $E = \{z : |z| < 1\}$. Let $U_\alpha$ denote the subclass of functions in $A_1$ which satisfy the conditions $\frac{f(z).f'(z)}{z} \neq 0$ and for $0 \leq \alpha \leq 1$, $\text{Re}\left[\alpha \frac{zf'(z)}{f(z)} + (1-\alpha)\frac{(zf'(z))'}{f'(z)}\right] > 0$. We are interested in determining the sharp upper bound for the functional $\left|a_2 a_4 - a_3^2\right|$ for the class $U_\alpha$ .

**MATHEMATICS SUBJECT CLASSIFICATION**: 30C45.

**KEYWORDS:** Analytic functions, univalent functions, starlike functions, convex functions, $\alpha$ convex functions, Bazilevic functions.

## I. INTRODUCTION:

Let $A_1$ be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

analytic in the unit disc $E = \{z : |z| < 1\}$.
S denotes the Class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.2}$$

analytic and univalent in $E = \{z : |z| < 1\}$.

Let $\gamma(p)$ be the class of functions of the form

$$\mathrm{P}(z)=1+p_1 z+p_2 z^2+p_3 z^3+\ldots \qquad (1.3)$$

analytic in the unit disc $E=\{z: |z|<1\}$ with $\operatorname{Re} P(z)>0$. Carathéodory [1] introduced the class $\gamma(p)$.

Noshiro [2] and Warschawiski [3] introduced the class of univalent functions

$$R=\{f\in A_1: \operatorname{Re} f'(z)>0, z\in E\}$$

(1.4)

known as N-W class of functions.

R and its subclasses were studied by several authors including Goel and Mehrok [11, 12].

$$S^*=\{ f\in A_1: \operatorname{Re}\frac{zf'(z)}{f(z)}>0, z\in E\}$$

(1.5)

is the class of starlike univalent functions .

$$\mathrm{K}=\{ f\in A_1: \operatorname{Re}\frac{(zf'(z))'}{f'(z)}>0, z\in E\}$$

(1.6)

is the class of convex univalent functions.

We introduce the class of $\alpha$ - convex functions defined as

$$U_\alpha=\left\{\begin{array}{l} f\in A_1: \dfrac{f(z)f'(z)}{z}\neq 0, \\ \operatorname{Re}\left[\alpha\dfrac{zf'(z)}{f(z)}+(1-\alpha)\dfrac{(zf'(z))'}{f'(z)}\right]>0, 0\le\alpha\le 1, z\in E\end{array}\right\}$$

(1.7)

For any real $\alpha$, these $\alpha$ - convex functions, are starlike in E; and for all $\alpha\le 0$, all $\alpha$ -convex functions are convex in E.

Again, we introduce the class of analytic functions

$$T_\alpha=\left\{ f\in A_1: \operatorname{Re}\left[\alpha f'(z)+(1-\alpha)\frac{(zf'(z))'}{f'(z)}\right]>0, 0\le\alpha\le 1, z\in E\right\} \qquad (1.8)$$

Bazilevic [4] introduced the following class of analytic univalent functions. For $\beta$ real, $\alpha>0$, $P(z)\in\gamma(p)$ and $g(z)\in S^*$

$$B(\alpha,\beta,P,g)=\left\{ f\in A_1:\left[(\alpha+i\beta)\int_0^z P(t)t^{\beta-1}g^{\alpha}(t)dt\right]^{\frac{1}{\alpha+t\beta}}\right\} \tag{1.9}$$

Taking $\beta=0$ and $g(z)\equiv z$ in (1.9), we get $B(\alpha,0,P,z)$ as the class of functions

$$B(\alpha,0,P,z)=\left\{ f\in A_1:\left[\alpha\int_0^z P(z)z^{\beta-1}dt\right]^{\frac{1}{\alpha}}\right\} \tag{1.10}$$

The class $B(\alpha,0,P,z)$ was studied by Singh [6] and El-Ashwah and Thomas [13].

$$B_\alpha=\left\{ f\in A_1:\operatorname{Re} f'(z)\left(\frac{f(z)}{z}\right)^{\alpha}>0,-1\le\alpha\le 0, z\in E\right\} \tag{1.11}$$

is also a subclass of Bazilevic functions.

## I. Preliminary Lemmas.

**Lemma 2.1[15]:** Let $\mathrm{P}(z)=1+p_1z+p_2z^2+p_3z^3+\ldots\in\gamma(p)$, then $|p_n|\le 2$ for all n (n=1,2,3...).

**Lemma 2.2[9]:** If $\mathrm{P}(z)=1+p_1z+p_2z^2+p_3z^3+\ldots\in\gamma(p)$, then

$2p_2=p_1^2+(4-p_1^2)x$ and

$4p_3=p_1^3+2p_1(4-p_1^2)x-p_1(4-p_1^2)x^2+2(4-p_1^2)(1-|x|^2)z$

for some x and z with $|x|\le 1,|z|\le 1$.

## II. Main Results.

**Theorem 3.1:** Let $f\in U_\alpha$, then

$$|a_2a_4-a_3^2|\le\frac{3(1-\alpha)(2-\alpha)^3}{(4-3\alpha)(3-2\alpha)^2(48-84\alpha+45\alpha^2-7\alpha^3)}+\frac{1}{(3-2\alpha)^2},-1<\alpha\le 0; \tag{3.1}$$

And $|a_2a_4-a_3^2|\le 1,\alpha=-1$ (3.2)

Results are sharp.

**Proof:** Since $f\in M_\alpha$, it follows that

$$\alpha\frac{zf'(z)}{f(z)}+(1-\alpha)\frac{(zf'(z))'}{f'(z)}=P(z) \tag{3.3}$$

Equating the coefficients in (3.3), it is easily established that

$$\left.\begin{aligned} a_2 &= \frac{p_1}{(2-\alpha)} \\ a_3 &= \frac{1}{2}\left[\frac{p_2}{(4-3\alpha)} + \frac{(4-3\alpha)p_1^2}{(3-2\alpha)(2-\alpha)^2}\right] \\ a_4 &= \frac{1}{(4-3\alpha)}\left[\frac{p_3}{3} + \frac{(6-5\alpha)p_1p_2}{2(2-\alpha)(3-2\alpha)} + \frac{(24-40\alpha+17\alpha^2)p_1^4}{6(3-2\alpha)(2-\alpha)^3}\right] \end{aligned}\right\} \tag{3.4}$$

System(3.4) yields

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{L(\alpha)}\left|\begin{array}{l} 16(3-2\alpha)^2(2-\alpha)^3 p_1p_3 + 24(3-2\alpha)(6-5\alpha)(2-\alpha)^2 p_1^2 p_2 \\ +8(3-2\alpha)(24-40\alpha+17\alpha^2)p_1^4 - 12(4-3\alpha)\left[(2-\alpha)^2 p_2 + (4-3\alpha)p_1^2\right]^2 \end{array}\right| \tag{3.5}$$

$$L(\alpha) = \frac{1}{48(4-3\alpha)(3-2\alpha)^2(2-\alpha)^4}\ . \tag{3.6}$$

Using lemma 2.2 in (3.5), we get

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{L(\alpha)}\left|\begin{array}{l} 4(3-2\alpha)^2(2-\alpha)^3 p_1\left[p_1^3 + 2p_1(4-p_1^2)x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)z\right] \\ +12(3-2\alpha)(6-5\alpha)(2-\alpha)^2 p_1^2\left(p_1^2 + (4-p_1^2)x\right) \\ +8(3-2\alpha)(24-40\alpha+17\alpha^2)p_1^4 - 3(4-3\alpha)\left[(12-10\alpha+\alpha^2)p_1^2 + (2-\alpha)^2(4-p_1^2)x\right]^2 \end{array}\right|$$

(3.7)

Replacing $p_1$ by $p \in [0,2]$, (3.7) takes the form

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{L(\alpha)}\left|\begin{array}{l} -\left[\begin{array}{l} -4(3-2\alpha)^2(2-\alpha)^3 - 12(3-2\alpha)(6-5\alpha)(1+\alpha)^2 \\ -8(3-2\alpha)(24-40\alpha+17\alpha^2) + 3(4-3\alpha)(12-10\alpha+\alpha^2)^2 \end{array}\right]p^4 \\ +\left[\begin{array}{l} 8(3-2\alpha)^2(2-\alpha)^3 + 12(3-2\alpha)(6-5\alpha)(2-\alpha)^2 \\ -6(4-3\alpha)(12-10\alpha+\alpha^2)(2-\alpha)^2 \end{array}\right]p^2(4-p^2)x \\ -(2-\alpha)^3(4-p^2)\left[12(2-\alpha)(4-3\alpha) + (12-18\alpha+7\alpha^2)\right]x^2 \\ +8(3-2\alpha)^2(2-\alpha)^3 p(4-p^2)(1-|x|^2)z \end{array}\right| \tag{3.8}$$

Applying triangular inequality to (3.8), we obtain

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{L(\alpha)}\left[\begin{array}{l}\begin{pmatrix}-4(3-2\alpha)^2(2-\alpha)^3-12(3-2\alpha)(6-5\alpha)(2-\alpha)^2\\-8(3-2\alpha)(24-40\alpha+17\alpha^2)+3(4-3\alpha)(12-10\alpha+\alpha^2)^2\end{pmatrix}p^4\\+\begin{pmatrix}8(3-2\alpha)^2(2-\alpha)^3+12(3-2\alpha)(6-5\alpha)(2-\alpha)^2\\-6(4-3\alpha)(12-10\alpha+\alpha^2)(2-\alpha)^2\end{pmatrix}p^2(4-p^2)|x|\\+(2-\alpha)^3(4-p^2)(2-p)(6(2-\alpha)(4-3\alpha)-(12-18\alpha+7\alpha^2))|x|^2\\+(8(3-2\alpha)^2(2-\alpha)^3)p(4-p^2)\end{array}\right] \tag{3.9}$$

$$=\frac{1}{L(\alpha)}F(\sigma),\sigma=|x|\le 1. \tag{3.10}$$

$F'(\sigma)>0$ and therefore $F(\sigma)$ is increasing in [0,1] and $F(\sigma)$ attains its maximum value at $|\sigma|=|x|=1$.

Putting $|x|=1$ in (3.9) , we have

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{L(\alpha)}\left[\begin{array}{l}\begin{pmatrix}-4(3-2\alpha)^2(2-\alpha)^3-12(3-2\alpha)(6-5\alpha)(2-\alpha)^2\\-8(3-2\alpha)(24-40\alpha+17\alpha^2)+3(4-3\alpha)(12-10\alpha+\alpha^2)^2\end{pmatrix}p^4\\+\begin{pmatrix}8(3-2\alpha)^2(2-\alpha)^3+12(3-2\alpha)(6-5\alpha)(2-\alpha)^2\\-6(4-3\alpha)(12-10\alpha+\alpha^2)(2-\alpha)^2\end{pmatrix}p^2(4-p^2)\\+(2-\alpha)^3(4-p^2)(12(2-\alpha)(4-3\alpha)+(12-18\alpha+7\alpha^2)p^2)\end{array}\right]$$

$$=\frac{1}{L(\alpha)}\left[\begin{array}{l}\begin{pmatrix}3(4-3\alpha)(12-10\alpha+\alpha^2)^2+6(4-3\alpha)(12-10\alpha+\alpha^2)(2-\alpha)^2-12(3-2\alpha)^2(2-\alpha)^3\\-24(3-2\alpha)(6-5\alpha)(2-\alpha)^2-8(3-2\alpha)(24-40\alpha+17\alpha^2)-(2-\alpha)^3(12-18\alpha+7\alpha^2)\end{pmatrix}p^4\\+4\begin{pmatrix}8(3-2\alpha)^2(2-\alpha)^3+12(3-2\alpha)(6-5\alpha)(2-\alpha)^2\\-6(4-3\alpha)(12-10\alpha+\alpha^2)(2-\alpha)^2+4(2-\alpha)^3(12-18\alpha+7\alpha^2)-12(4-3\alpha)(2-\alpha)^4\\+4(2-\alpha)^3(11-17\alpha+7\alpha^2)\end{pmatrix}p^2\\+48(4-3\alpha)(2-\alpha)^4\end{array}\right]$$

$$\left.\begin{array}{l}=\frac{1}{L(\alpha)}\left[-A(\alpha)p^4+B(\alpha)p^2+48(4-3\alpha)(2-\alpha)^4\right]\\=\frac{1}{L(\alpha)}G(p)\end{array}\right] \tag{3.11}$$

where $A(\alpha)=4(1-\alpha)(2-\alpha)(-7\alpha^3+45\alpha^2-36\alpha+38)$, (3.12) and $B(\alpha)=48(1-\alpha)(2-\alpha)^4$. (3.13)

$G'(p)=-4A(\alpha)p^3+2B(\alpha)p=0$ (3.14)

which implies that p=0 or $p^2=\frac{B(\alpha)}{2A(\alpha)}$.

p=0 does not give maximum value and is rejected.

$$p^2=\frac{B(\alpha)}{2A(\alpha)}=\frac{6(2-\alpha)^3}{(48-84\alpha+45\alpha^2+7\alpha^3)}\ , \quad (\alpha\neq 1) .(3.15)$$

gives the maximum value of $G(p)$.

Putting the value of $p^2$ from (3.15) in (3.11) , we get

$$\left|a_2a_4-a_3^2\right|\leq\frac{1}{L(\alpha)}\left[\frac{B^2(\alpha)}{4A(\alpha)}+48(4-3\alpha)(2-\alpha)^4\right]\ ,\ \alpha\neq 1$$

(3.16)

Substituting the values from (3.6), (3.12) and (3.13) in (3.16) , the bound (3.1) follows.
Consider the case $\alpha=0$ . In this case, $A(\alpha)=0, B(\alpha)=0$ and $L(\alpha)=48$ .

Putting these values in (3.11) , we get $\left|a_2a_4-a_3^2\right|\leq 1$ .

Result (3.1) is sharp for $p_1=\sqrt{\frac{6(2-\alpha)^3}{(48-84\alpha+45\alpha^2+7\alpha^3)}}$ , $p_2=-1$ and $p_3$ obtained from (3.5).

Result (3.2) is sharp for $p_1=0, p_2=-1$ and $p_3=-2$.

**Remark 3.1:** Taking $\alpha=1$ in (3.1) , we get $\left|a_2a_4-a_3^2\right|\leq\frac{1}{8}$ , a result due to Jantengetal.[15].

**Remark 3.2:** Result (3.2) is also due to Jantengetal.[15].

**Theorem 3.2:** Let $f\in T_\alpha$, then

$$\left|a_2a_4-a_3^2\right|\leq\begin{cases}\frac{(12-17\alpha)^2}{144(3-2\alpha)(24-22\alpha-5\alpha^2+4\alpha^3)}+\frac{4}{9(2-\alpha)^2}\ if\ 0\leq\alpha\leq\frac{12}{17}\\ \frac{4}{9(2-\alpha)^2}\ if\ \frac{12}{17}\leq\alpha\leq 1\end{cases}. \qquad (3.17)$$

Results are sharp.

**Proof:** Since $f\in T_\alpha$, therefore by definition

$$\alpha f'(z)+(1-\alpha)\frac{(zf'(z))'}{f'(z)}=P(z)$$

Identification of terms in the above equation yields

$$\left.\begin{aligned}a_2&=\frac{p_1}{2}\\ a_3&=\frac{\left(p_2+(1-\alpha)p_1^2\right)}{3(2-\alpha)}\\ a_4&=\frac{1}{4(3-2\alpha)}\left[p_3+\frac{3(1-\alpha)p_1p_2}{2-\alpha}+\frac{(1-\alpha)(1-2\alpha)p_1^3}{2-\alpha}\right]\end{aligned}\right\}(3.18)$$

From (3.19), we obtain

$$\left|a_2a_4-a_3^2\right|=\frac{1}{K(\alpha)}\left|\begin{matrix}9(2-\alpha)^2p_1(4p_3)-54(1-\alpha)(2-\alpha)p_1^2(2p_2)\\ -36(1-\alpha)(1-2\alpha)\alpha p_1^4-8(3-2\alpha)\left(2p_2+2(1-\alpha)p_1^2\right)^2\end{matrix}\right|, \qquad (3.19)$$

Where $K(\alpha)=\dfrac{1}{288(3-2\alpha)(2-\alpha)^2}$.

(3.20)

By lemma 2.2, we get

$$\left|a_2a_4-a_3^2\right|=\frac{1}{K(\alpha)}\left|\begin{matrix}9(2-\alpha)^2p_1\left[p_1^3+2p_1(4-p_1^2)x-p_1(4-p_1^2)x^2+2(4-p_1^2)(1-|x|^2)z\right]\\ +54(2-\alpha)p_1^2\left[p_1^2+(4-p_1^2)x\right]-8(3-2\alpha)\left[(3-2\alpha)p_1^2+(4-p_1^2)x\right]^2\\ +36(1-\alpha)(3-2\alpha)(1+\alpha)p_1^4\end{matrix}\right| \qquad (3.21)$$

$|x|\le 1$ and $|z|\le 1$. Changing $p_1$ to $p\in[0,2]$, (3.22) takes the form

$$\left|a_2a_4-a_3^2\right|=\frac{1}{K(\alpha)}\left|\begin{matrix}-\alpha(18-27\alpha+8\alpha^2)p^4+2(18-21\alpha+4\alpha^2)p^2(4-p^2)x\\ -(4-p^2)\left[32(3-2\alpha)+(12-20\alpha+9\alpha^2)p^2\right]\\ +18(2-\alpha)^2p(4-p^2)(1-|x|^2)z\end{matrix}\right|$$

$$\le\frac{1}{C(\alpha)}\left[\begin{matrix}-\alpha(18-27\alpha+8\alpha^2)p^4+2(18-21\alpha+4\alpha^2)p^2(4-p^2)\sigma\\ +(4-p^2)(2-p)\left[16(3-2\alpha)-(12-20\alpha+9\alpha^2)p^2\right]\sigma^2\\ +18(2-\alpha)^2p(4-p^2)\end{matrix}\right]=\frac{1}{K(\alpha)}F(\sigma), \qquad (3.22)$$

Where $\sigma=|x|\le 1$.

$F'(\sigma)>0$ and therefore $F(\sigma)$ is increasing in [0,1] and maximum $F(\sigma)=F(1)$.

Putting the value $\sigma=1$ in (3.23), we arrive at

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{K(\alpha)}\left[\begin{matrix}-\alpha(18-27\alpha+8\alpha^2)p^4+2(18-21\alpha+4\alpha^2)p^2(4-p^2)\\ +(4-p^2)\left(32(3-2\alpha)+(12-20\alpha+9\alpha^2)p^2\right)\end{matrix}\right]$$

$$= \frac{1}{K(\alpha)}\left[\begin{array}{l}\left(-\alpha\left(18-27\alpha+8\alpha^2\right)-2\left(18-21\alpha+4\alpha^2\right)-\left(12-20\alpha+9\alpha^2\right)\right)p^4 \\ +\left[8\left(15-15\alpha+\alpha^2\right)-32(3-2\alpha)+4\left(12-20\alpha+9\alpha^2\right)\right]p^2+128(3-2\alpha)\end{array}\right]$$

$$\left.\begin{array}{l} = \frac{1}{K(\alpha)}\left[-A(\alpha)p^4+B(\alpha)p^2+128(3-2\alpha)\right] \\ = \frac{1}{K(\alpha)}G(p)\end{array}\right] \qquad (3.23)$$

Where $A(\alpha)=2\left(19-7\alpha-20\alpha^2+9\alpha^3\right)>0$ in [0, 1]
(3.24)
And $B(\alpha)=4(2-\alpha)(12-17\alpha)$.
(3.25)

Case I: $\frac{12}{17}\le\alpha\le 1$ so that $B(\alpha)\le 0$.

$G'(p)<0$ and $G(p)$ attains its maximum value at $p=0$.

From (3.21) and (3.24) it follows that $\left|a_2a_4-a_3^2\right|\le\frac{4}{9(2-\alpha)^2}$.

Result is sharp for $p_1=0, p_2=-1$ and $p_3=-2$.

Case II: $0\le\alpha\le\frac{12}{17}$ so that $B(\alpha)>0$.

$G'(p)=-4A(\alpha)p^3+2B(\alpha)p=0$ which implies that

$p=0$ or $p^2=\frac{B(\alpha)}{2A(\alpha)}=\frac{(2-\alpha)(12-17\alpha)}{\left(24-22\alpha-5\alpha^2+4\alpha^3\right)}$
(3.26)

$p=0$ does not give the maximum value and is rejected.

Substituting the value of $p^2$ from (3.27) in (3.24), we conclude that

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{K(\alpha)}\left[\frac{B^2(\alpha)}{4A(\alpha)}+128(3-2\alpha)\right] \qquad (3.27)$$

Putting the values from (3.20), (3.24) and (3.25) in (3.27), result (3.17) follows.

Equality sign in (3.18) holds for $p_1=\sqrt{\frac{(2-\alpha)(12-17\alpha)}{\left(24-22\alpha-5\alpha^2+4\alpha^3\right)}}, p_2=-1$ and $p_3$ obtained from (3.19).

**Remark 3.3:** Letting $\alpha=1$ in (3.17) , we get

$\left|a_2a_4 - a_3^2\right| \le \frac{4}{9}$, a result proved by Jantengetal.[7] for the class R.

**Remark 3.4:** Letting $\alpha = 0$ in (3.18), it follows that

$\left|a_2a_4 - a_3^2\right| \le \frac{1}{8}$, result proved by Jantengetal.[8] for the class K

**Theorem 3.3:** Let $f \in B_\alpha$, then $\left|a_2a_4 - a_3^2\right| \le \frac{4}{(\alpha+3)^2}$.

(3.28)

The result is sharp.

**Proof:** Since $f \in B_\alpha$, therefore it follows that

$$f'(z)\left(\frac{f(z)}{z}\right)^\alpha = P(z), (0 \le \alpha \le 1). \tag{3.29}$$

Equating the coefficients in (3.30), we get

$$\left.\begin{aligned} a_2 &= \frac{p_1}{(\alpha+2)} \\ a_3 &= \frac{p_2}{(\alpha+3)} - \frac{\alpha p_1^2}{2(\alpha+2)^2} \\ a_4 &= \frac{p_3}{(\alpha+4)} - \frac{\alpha p_1 p_2}{(\alpha+2)(\alpha+3)} + \frac{\alpha(1+2\alpha)p_1^3}{6(\alpha+2)^3} \end{aligned}\right\} (3.30)$$

(3.31) gives

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{C(\alpha)}\left|\begin{aligned} &3(\alpha+3)^2(\alpha+2)^3 p_1(4p_3) - 6\alpha(\alpha+3)(\alpha+2)^2 p_1(2p_2) \\ &+2\alpha(1+2\alpha)(\alpha+4)(\alpha+3)^2 p_1^4 - 3(\alpha+4)\left[2(2+\alpha)p_2 - \alpha(\alpha+3)p_1^2\right]^2 \end{aligned}\right| \tag{3.31}$$

Where $N(\alpha) = \frac{1}{12(\alpha+4)(\alpha+3)^2(\alpha+2)^4}$.

(3.32)

Using lemma 2.2 in (3.32), we obtain

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{N(\alpha)}\left|\begin{aligned} &3(\alpha+3)^2(\alpha+2)^3 p_1\left[p_1^3 + 2p_1(4-p_1^2)x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)z\right] \\ &-6\alpha(\alpha+3)(\alpha+4)(\alpha+2)^2 p_1^2\left[p_1^2 + (4-p_1^2)x\right] \\ &+2\alpha(1+2\alpha)(\alpha+4)(\alpha+3)^2 p_1^4 - 3(\alpha+4)\left[(\alpha+4)p_1^2 + (2+\alpha)^2(4-p_1^2)x\right]^2 \end{aligned}\right| (3.33)$$

Changing $p_1$ to $p \in [0,2]$, (3.34) takes the form

$$\left|a_2a_4-a_3^2\right|=\frac{1}{N(\alpha)}\left|\begin{matrix}\left[\begin{matrix}3(\alpha+3)^2(\alpha+2)^3-6\alpha(\alpha+3)(\alpha+4)(\alpha+2)^2\\+2\alpha(1+2\alpha)(\alpha+4)(\alpha+3)^2-3(\alpha+4)^3\end{matrix}\right]p^4\\+\left[\begin{matrix}6(\alpha+3)^2(\alpha+2)^3-6\alpha(\alpha+3)(\alpha+4)(\alpha+2)^2\\-6(\alpha+2)^2(\alpha+4)^2\end{matrix}\right]p^2\left(4-p^2\right)x\\-3(\alpha+2)^3\left(4-p^2\right)\left[4(\alpha+2)(\alpha+4)+p^2\right]x^2\\+6(\alpha+3)^2(\alpha+2)^3p\left(4-p^2\right)\left(1-|x|^2\right)z\end{matrix}\right| \quad (3.34)$$

Coefficient of $p^4$ in (3.35) changes from negative to positive in [0, 1] and therefore there must exist $\alpha_0$ in (0, 1) so that coefficient of $p^4$ is negative for $0\le\alpha<\alpha_0$ and positive for $\alpha_0\le\alpha\le1$.

Case I: $0\le\alpha<\alpha_0$, from (3.35) it follows that

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{N(\alpha)}\left[\begin{matrix}\left(\begin{matrix}3(\alpha+4)^3+6\alpha(\alpha+3)(\alpha+4)(\alpha+2)^2-3(\alpha+3)^2(\alpha+2)^3\\-2\alpha(1+2\alpha)(\alpha+4)(\alpha+3)^2\end{matrix}\right)p^4\\+6\left(\begin{matrix}(\alpha+3)^2(\alpha+2)^3-\alpha(\alpha+3)(\alpha+4)(\alpha+2)^2\\-(\alpha+2)^2(\alpha+4)^2\end{matrix}\right)p^2\left(4-p^2\right)x\\+3(\alpha+2)^3\left(4-p^2\right)(2-p)\left[2(\alpha+2)(\alpha+4)-p\right]x^2\\+6(\alpha+3)^2(\alpha+2)^3p\left(4-p^2\right)\end{matrix}\right]$$

$$=\frac{1}{N(\alpha)}F(\sigma),\sigma=|x|\le1\ . \quad (3.35)$$

$F'(\sigma)>0$ and therefore $F(\sigma)$ is increasing in [0,1] and $F(\sigma)$ will attain its maximum value at $\sigma=|x|=1$.

Putting $\sigma=|x|=1$ in (3.36), we get

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{N(\alpha)}\left[\begin{matrix}\left(\begin{matrix}3(\alpha+4)^3+6(\alpha+4)^2(\alpha+2)^2+12\alpha(\alpha+3)(\alpha+4)(\alpha+2)^2\\-9(\alpha+3)^2(\alpha+2)^3-2\alpha(1+2\alpha)(\alpha+4)(\alpha+3)^2-3(\alpha+2)^3\end{matrix}\right)p^4\\+12\left(\begin{matrix}2(\alpha+3)^2(\alpha+2)^3-2\alpha(\alpha+3)(\alpha+4)(\alpha+2)^2\\-2(\alpha+2)^2(\alpha+4)^2-(\alpha+2)^3\left[(\alpha+2)(\alpha+4)+1\right]\end{matrix}\right)p^2\left(4-p^2\right)\\+48(\alpha+1)(\alpha+4)(\alpha+2)^4\end{matrix}\right]$$

$$\left.\begin{array}{l}= \dfrac{1}{N(\alpha)}\left[-A_1(\alpha)p^4 - B(\alpha)p^2 + 48(4+3\alpha)(2+\alpha)^4\right] \\ = \dfrac{1}{N(\alpha)}G(p)\end{array}\right]. \qquad (3.36)$$

Where $\left.\begin{array}{l}A_1(\alpha) = (\alpha+1)(\alpha+2)\left(\alpha^3+9\alpha^2+36\alpha+27\right) > 0 \\ B(\alpha) = 12(\alpha+1)(\alpha+5)(\alpha+2)^3 > 0\end{array}\right].$ (3.37)

$G'(p) < 0$, $G(p)$ is decreasing in [0, 2] and maximum $G(p) = G(0) = 48(4+3\alpha)(\alpha+2)^4$.
From (3.37) and (3.33), it follows that

$\left|a_2a_4 - a_3^2\right| \le \dfrac{4}{(\alpha+3)^2}$.

Case II: $\alpha_0 \le \alpha \le 0$, proceeding as in case I, from (3.35) , we get

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{N(\alpha)}\left[\begin{array}{l}\left[\begin{array}{l}\left(6(\alpha+3)^2(\alpha+2)^2 + 2\alpha(2\alpha-3)(\alpha+4)(\alpha+3)^2\right) \\ -3\left((\alpha+3)^2(\alpha+2)^3 - (\alpha+4)^3 - (\alpha+2)^3\right)\end{array}\right]p^4 \\ -B(\alpha)p^2 + 48(4+3\alpha)(2+\alpha)^4\end{array}\right].$$

Where $B(\alpha)$ is given by (3.38))

$$= \frac{1}{N(\alpha)}\left[-A_2(\alpha)p^4 - B(\alpha)p^2 + 48(4+3\alpha)(2+\alpha)^4\right]. \qquad (3.38)$$

where $A_2(\alpha) = \left[53 + 15\alpha - 51\alpha^2 - 42\alpha^3 - 12\alpha^4 - \alpha^5\right] > 0$.

As discussed in case I, $\left|a_2a_4 - a_3^2\right| \le \dfrac{4}{(\alpha+3)^2}$.

Combining both the cases, proof of the theorem is complete.
Result (3.29) is sharp for $p_1 = 0$ , $p_2 = -1$ and $p_3 = -2$.

**Remark 3.5:** Putting $\alpha = -1$in (3.29) , we have
$\left|a_2a_4 - a_3^2\right| \le 1$, a result established by Jantengetal.[15] for the class $S^*$.

**Remark 3.6:** If we put $\alpha = 0$in (3.29), we get
$\left|a_2a_4 - a_3^2\right| \le \dfrac{4}{9}$, a result established by Jantengetal.[14] for the class R.

## References

[1] C. Carathéodory, Über Den VariabilitatsbErecöch Der FourieŕSchenKonstanten Von Positivenenharmonischen-Funktionan, RendicontiPalerno, 32(1911), 193-217.

[2] K.Noshiro, On the theory of schlichtfunctions,J. Fac. Soc. Hokkaido univ.Ser.1,2(1934-1935), 129-155.

[3] S. Warchawski, on the higher derivative at the boundary in conformal mapping, Trasaction Amer. Math.Soc., 38(1935), 310-340.

[4] I.E.Bazilevic, On the case of integrability in quadrature of the Lowner equation, Mat Sb. 37(1958), 471-476.

[5] P.T.Mocanu, Unepropritédeconvexité generalise dans la théorie de la representation conforme,Mathematica (CLUJ), 34 (1969), 127-133.

[6] R.Singh, On Bazilevic functions, Proceeding Amer. Math.Soc., 38(1973), 261-263.

[7] S.S.Miller ,P.T.Mocanu and M.O.Reade, All $\alpha$ -convex functions are univalent and starlike, Proceeding Amer. Math.Soc.,37(1973), 553-554.

[8] Ch. Pommerenke, Univalent functions, Vandenhoeck and Ruprecht, Gottingen, 1975.

[9] S.Hassoon,Al-Amiri and M.O.Reade, On A linear combination of some expressions in the theory of the univalent functions, Monatshafte fur Math.,80(1975),257-264.

[10] R.J.Libera and E.J.Zlotkiewicz, Early coefficients of the inverse of a regular convex function, Proceeding Amer. Math.Soc., 85(1982), 225-230.

[11] R.M.Goel and B.S.Mehrok, A subclass of univalent functions, Houston J.Math., 8(1982), 343-357.

[12] R. M. Goel, Beant Singh Mehrok, A subclass of univalent functions, J.Austral.Math.Soc. (Series A),35(1983), 1-17.

[13] R.M.El-Ashwah and D.K.Thomas, Growth results for a subclass of Bazilevic functions, Inter J.Math.ana Math Sci, 8(1985), 785-792.

[14] A.Janteng ,S.A.Halim and M.Darus, Coefficient inequality for a function whoose derivative has a positive real part, J.Ineq. in pure and appl. Math., 7(2006), 1-4.

[15] AiniJanteng, Hankel determinant for starlike and convex functions, Int. J.Math.Anal., 1(2007), 619-625.