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Comparative Analy Sis Of Ordinary And Fractional Differential Equations In Modeling Physical Phenomena

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Abstract: This study provides a thorough comparison between fractional differential equations (FDEs) and ordinary differential equations (ODEs). In classical physics, ordinary differential equations which contain derivatives of integer order have been extensively employed to explain a variety of dynamic systems, including motion, heat conduction, and wave propagation. Nevertheless, these models frequently take local interactions and homogeneity for granted, possibly ignoring intricate memory effects and spatial interactions that are present in a lot of real-world systems. Fractional differential equations are particularly useful in capturing anomalous diffusion, viscoelastic behavior, and other non-local dynamics because they provide a more generalized framework that takes these complexities into account.

Fractional differential equations incorporate derivatives of non-integer order. Through an analysis of the precision, computational effectiveness, and practicality of ODEs and FDEs in various scenarios, this research identifies the key use cases for fractional models, particularly in long-term memory or geographically diverse systems. The analysis highlights the significance of selecting the right mathematical model based on the particulars of the physical event under study, pointing out that although FDEs provide a fuller, more nuanced representation of complex systems, ODEs are more straightforward and well-established.

Keywords: Ordinary Differential Equations, Fractional Differential Equations, Modelling, Physical Phenomena.

1. INTRODUCTION

Differential equations are a fundamental tool in the mathematical description of changes in systems in a wide range of fields, including physics, engineering, biology, and economics. They are used extensively in the modelling of physical processes. The cornerstone of this strategy has been ordinary differential equations (ODEs), in particular, which provide a triedand-true methodology to depict the evolution of systems over time. When modelling processes that vary continuously with regard to one or more independent variables, these equations play a crucial role in connecting a function and its derivatives. ODEs have been useful in explaining a variety of scientific phenomena, including population dynamics, electrical circuits, radioactive material decay, planet motion, and more.

But when dealing with complex systems that display memory effects, non-local interactions, or anomalous diffusion processes - where the system's current state is influenced by a history of states over time as well as its immediate past-the limitations of ODEs become evident. Since traditional ODEs are unable to fully describe such occurrences, fractional differential equations (FDEs) have emerged as a potent substitute. By adding derivatives of non-integer orders, FDEs extend ODEs and add a fractional dimension to the analysis. The modelling of systems with memory and hereditary characteristics is made possible by this fractional dimension. These systems are common in many domains, such as complex biological systems, anomalous transport in porous media, and viscoelastic materials.

A greater comprehension of the advantages and disadvantages of each modelling technique is possible through the comparison of ODEs and FDEs. ODEs function well in situations where the interactions between variables are instantaneous and obvious, while FDEs provide a more flexible framework that can consider the complex dynamics of systems that have non-local dependencies. For example, FDEs offer a more realistic representation than ODEs for modelling viscoelastic materials, where stress and strain connections rely on the history of deformation. ODEs would need to make extra, frequently difficult assumptions to account for such effects.

Moreover, the adaptability of FDEs is demonstrated by their application in the description of anomalous diffusion, in which particles propagate in a medium at a rate distinct from the classical Brownian motion. Conventional models that employ ODEs usually fall short of describing the super-diffusive or sub-diffusive behavior that is found in these kinds of systems. On the other hand, FDEs provide a natural and effective technique to represent these phenomena because of their intrinsic ability to explain processes with memory.

When comparing fractional and ordinary differential equations, it becomes clear that FDEs are required in some complicated situations where regular ODEs are insufficient. FDEs offer a deeper and more accurate representation of many physical processes by expanding on the traditional concept of derivatives and incorporating memory effects into the mathematical models. This research provides fresh insights into the behavior of complex systems that are memory-dependent and non-local, and it also emphasizes how important it is to choose the right mathematical tool for the job at hand.

2. LITERATURE REVIEW

Dubey et al. (2023) examine local fractional partial differential equations (FPDEs) in the context of the physical sciences, including their analysis and fractal dynamics. Their research makes a substantial contribution to our knowledge of how FPDEs can be used to simulate complicated events with fractal features. The authors investigate local FPDE solutions and their applications in fractal-behaving physical systems using a variety of analytical methods. The study emphasizes how FPDEs can be used to capture minute aspects of physical processes that conventional differential equations could miss. Researchers who wish to model systems with temporal or spatial abnormalities may find this study useful since it offers a comprehensive framework for examining the fractal structure of physical occurrences.

Ganie, AlBaidani, and Khan (2023) utilizing a unique transform approach, offer a comparative study of fractional partial differential equations (FPDEs). The efficiency of various fractional transforms in solving FPDEs is the main topic of this study. The authors examine the applicability and efficiency of several transformation approaches in managing FPDEs across different circumstances. The research sheds light on the advantages and disadvantages of each transform approach, making it a useful tool for choosing the best approaches depending on the particulars of the FPDE under consideration. The advancement of FPDE applications in physics, engineering, and applied mathematics depends on this comparative examination.

Ismail et al. (2020) describe the approximation and analysis of solutions to fractional physical phenomena using the fractional residual power series method (FRPSM). The goal of this approach is to improve the ability to handle fractional systems that occur in physical situations by offering both analytical and approximation solutions for FPDEs. The FRPSM provides a methodical way to arrive at solutions and evaluate their precision. Using a number of example studies, the study shows how FRPSM may be applied to effectively capture the behavior of fractional systems. This work provides a useful tool for solving complicated fractional models in the physical sciences and is important for researchers looking for reliable ways to handle fractional differential equations.

Jhangeer et al. (2022) examine how water waves propagate using fractional operators and provide a comparative analysis to see how fractional calculus might improve wave phenomena modelling. In contrast to classical models, the authors examine several fractional operators and their effects on the properties of wave propagation. The study intends to solve shortcomings in conventional wave models and provide more precise descriptions of water wave behavior under different conditions by using fractional calculus. This work is important for refining oceanic models and comprehending fractional effects-influenced complicated wave dynamics.

Kumam et al. (2022) perform an extensive investigation into the impacts on the fluid flow caused by the unstable magneto hydrodynamic (MHD) radioactive flow of a rate-type fluid using fractional calculus. To assess how well various fractional models, capture the subtleties of MHD radiative flows, the research presents a comparative examination of them. The authors examine how fractional differential equations affect the fluid's behavior under different circumstances using simulation-based techniques. This work advances our knowledge of complex fluid dynamics in radiative and magnetic fields and provides insights into the use of fractional calculus in MHD issues.

Mofarreh et al. (2023) provide a comparative study of the FO-FPE, or fractional-order Fokker-Planck equation, with an emphasis on how it can be used to represent different types of stochastic processes. Examining the distinctions between classical and fractional-order Fokker-Planck equations, the paper emphasizes the impact of fractional calculus on the description of diffusion processes and probability densities. The paper offers a thorough evaluation of the benefits and drawbacks of utilizing fractional-order models in stochastic systems through a comparison of analytical and numerical solutions. Researchers looking to improve fractional calculus-based stochastic process modeling will find this study useful.

3. MATHEMATICAL AND COMPUTATIONAL CHALLENGES

3.1. Analytical Solutions and Complexity of ODEs and FDEs

Due to the fact that they offer precise formulas that explain the behavior of physical systems, analytical solutions to Ordinary Differential Equations (ODEs) are highly desired. A analytical solutions for ODEs are available for a wide range of classical problems, including population dynamics, electrical circuits, and basic harmonic oscillators. One illustration might be the first-order linear ODE:

$$\frac{dy}{dt} + p(t)y = q(t) \ldots..(1)$$

can be resolved with the help of an integrating component, providing the overall solution:

$$y(t) = e^{-\int p(t)dt}\left(\int q(t)e^{\int p(t)dt}dt + C\right) \ldots\ldots(2)$$

where the constant C is arbitrary. One of ODEs' advantages in many real-world situations is their solutions' clarity and simplicity. However, since fractional derivatives are non-local, solving analytical solutions to Fractional Differential Equations (FDEs) becomes much more difficult. For example, the overall solution of a basic FDE:

$$D^{\alpha}y(t) + \lambda y(t) = 0 \ldots.(3)$$

With specific functions like the Mittag-Leffler function, where λ is a constant and Dα is the fractional derivative:

$y(t) = E_{\alpha}(-\lambda t^{\alpha})$…..(4)

The exponential function is generalized as the Mittag-Leffler function E $\alpha(z)$, which lacks the ease of solution found in ODEs. FDEs are difficult to work with because of their analytical complexity, particularly for more complex systems for which closed-form solutions might not exist or would be difficult to understand.

3.2. Numerical Methods for Solving FDEs

Numerical approaches are essential for solving FDEs because it is difficult to acquire analytical answers. Because fractional derivatives have a memory effect, numerical methods for solving FDEs are typically more complicated than those for solving ODEs. The Grunwald-Letnikov approach is a popular technique for solving FDEs; it discretizes the fractional derivative as:

$D^{\alpha} y(t_n) \approx \frac{1}{h^{\alpha}} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} y(t_{n-k})$…..(5)

where $tn = nh$ and h is the time step. The memory effect is reflected in this method, which is conceptually simple but computationally demanding because it needs to calculate every state at every time step.

The fractional Adams-Bashforth-Moulton approach, an expansion of the traditional Adams Bashforth-Moulton method for ODEs, is another well-liked technique. In order to account for the fractional character of the problem, this predictor-corrector method uses an iterative process to approximate the answer at each time step.

Although these techniques work well, they frequently call for higher processing power and fewer time steps than those required solving ODEs. This increases the difficulty of finding numerical solutions for FDEs, particularly in the case of systems with long memory effects.

All things considered, FDEs are effective tools for modelling memory-dependent systems, but they also present mathematical and computational hurdles that need to be carefully considered, especially with regard to analytical complexity and numerical solution techniques.

4. COMPARATIVE ANALYSIS OF ODES AND FDES

4.1. Strengths and Limitations of ODEs in Physical Phenomena

ODEs, or ordinary differential equations, are frequently employed to simulate a wide range of physical events because of their capacity to depict dynamic systems. ODEs' main advantages are their mathematical foundations and ease of use, which make them ideal for systems with deterministic and time-invariant characteristics. One way to express Newton's second law of motion, $F = ma$, is as a second-order ODE:

$$m\frac{d^2x(t)}{dt^2} = F(x,t).....(6)$$

Here, the object's position at time t is denoted by $x(t)$, and the force acting on it is represented by $F(x,t)$. These kinds of ODEs work especially well when modelling systems in which the future state is only dependent on the present state and not on the system's past.

However, when dealing with systems that show memory effects or when the dynamics of the past affect the dynamics of the present, the limitations of ODEs become evident. ODEs may not be appropriate for complicated systems with long-term dependencies, such as viscoelastic materials, anomalous diffusion, or specific biological processes, since they presume that the dependent variable's rate of change depends only on its current state.

4.2. Advantages of FDEs in Modeling Memory-Dependent Systems

The idea of ODEs is expanded upon by fractional differential equations (FDEs), which include derivatives of non-integer (fractional) orders. The system's whole history influences the current state, and this expansion enables FDEs to simulate systems with memory effects. Caputo or Riemann-Liouville definitions are commonly employed to define the fractional derivative. Using order α as an example, the Caputo fractional derivative is provided by:

$$^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau,\ (n-1<\alpha<n)....(7)$$

where $f(n)(\tau)$ is the n -th derivative of $f(\tau)$, and $\Gamma(\cdot)$ is the Gamma function. FDEs are especially useful for systems like viscoelastic materials, where stress depends on the entire strain history, or anomalous diffusion processes, where the mean squared displacement is not linear with time but rather exhibits a power-law behavior. The fractional derivative offers a mathematical tool to model the memory-dependent behavior in various systems.

Compared to typical integer-order models, FDEs have the benefit of being able to represent complex dynamics with fewer parameters. Because of this, FDEs are effective in explaining occurrences in the actual world that show non-locality and history-dependent effects, giving systems with memory a more accurate depiction.

4.3. Case Studies Illustrating Differences in Modelling Approaches

To demonstrate the distinctions between ODEs and FDEs in modelling, have a look at the diffusion process as an example. An ODE known as the classical diffusion equation is provided by:

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} \ldots..(8)$$

When D is the diffusion coefficient and $u(x,t)$ is the concentration of a material at position xxx and time t . The normal diffusion model represented by this equation has a mean squared displacement of particles that is linear in time, or $\langle x2(\mathrm{t})\rangle \sim \mathrm{t}$.

An FDE is more suited, nevertheless, in systems that exhibit anomalous diffusion, where the mean squared displacement follows a power-law, $\langle \mathrm{x2(t)}\rangle \sim \mathrm{t\alpha}$ (where $\alpha > 1$ for super diffusion and $0 < \alpha < 1$ for sub diffusion). One way to express the fractional diffusion equation is as:

$$\frac{\partial^\alpha \mathrm{u(x,t)}}{\partial \mathrm{t}^\alpha} = \mathrm{D}_\alpha \frac{\partial^2 \mathrm{u(x,t)}}{\partial \mathrm{x}^2} \ldots..(9)$$

Here, $\frac{\partial^\alpha}{\partial \mathrm{t}^\alpha}$ represents the time-dependent Caputo fractional derivative, and Dα is the generalized diffusion coefficient. A typical ODE is unable to adequately characterize the sub diffusive or super diffusive behavior that this FDE captures by considering the historical influence of the system's states. There is also the case of viscoelasticity. Compared to the straightforward ODE-based Hooke's law used for elastic materials, an FDE that considers the material's memory more precisely models the stress-strain relationship in a viscoelastic material. The stress $\sigma(\mathrm{t})$ in this situation can be stated as follows:

$$\sigma(\mathrm{t}) = \mathrm{E}\frac{\mathrm{d}^\alpha \epsilon(\mathrm{t})}{\mathrm{dt}^\alpha} \ldots..(10)$$

where the material's viscoelastic qualities are reflected by α, a fractional order, and the strain, represented by $\sigma(\mathrm{t})$. The history-dependent behavior of stress in viscoelastic materials is captured by this equation, which is not captured by a traditional ODE model. These case studies show how ODEs are inadequate for adequately modelling complicated physical events with memory effects and emphasize the need for FDEs instead.

5. CONCLUSION

Ordinary differential equations (ODEs) have long been the foundation of mathematical modelling in classical physics, but when dealing with complex systems that exhibit memory effects, spatial heterogeneity, and non-local interactions, the limitations of ODEs become evident. This is the conclusion of a comparative analysis between ODEs and fractional differential equations (FDEs) in the modelling of physical phenomena. A more accurate and thorough representation of such systems is made possible by fractional differential equations, a potent extension of standard ODEs that can express derivatives of non-integer order. FDEs are especially useful for describing phenomena such as anomalous diffusion, viscoelasticity, and others in which the current behavior of the system is strongly influenced by past states or spatial interactions. But the complexity of FDEs increases with difficulty, requiring more sophisticated numerical techniques and more demanding processing resources. ODEs may be sufficient for simpler, well-localized processes, but FDEs offer a more flexible and realistic modelling approach for systems with complex dynamics. Consequently, the decision between ODEs and FDEs should be made based on the particular requirements of the physical system under study. This investigation highlights the dynamic field of differential equation modelling, where integrating fractional calculus is crucial to improving our comprehension of intricate physical phenomena.

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