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Coefficient Inequality For A New Subclass Of Starlike Function

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Abstract- The aim of the present paper is to investigate a certain subclass $S^*(A,B)(p)$ of starlike function and obtain the sharp upper bound of the functional $\left|a_3-\mu a_2^2\right|$ for the analytic function

$$f(z)=z+a_2z^2+a_3z^3+\cdots, \qquad |z|<1$$

belonging to this subclass of starlike function.

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1 INTRODUCTION

Let A denote the class of functions of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots \qquad (1)$$

Which are analytic in the open unit disc $U = z : z \in C; |z| < 1$ and let S denote the class of functions in A that are univalent in U.

In 1916, for the functions $f(z)\in S$, Bieber Bach [3] proved the result $|a_2| \le 2$. In 1923, for the same functions, Lowner [15] proved that $|a_3| \le 3$. With these results $|a_2| \le 2$ and $|a_3| \le 3$, for the class S it was very easy to draw out the relation between a_3 and a_2^2. With the help of Lowner's method , Fekete and Szego [6] proved the following well known result

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} 3-4\mu & if \quad \mu \le 0 \\ 1+\exp\left(\dfrac{-2\mu}{1-\mu}\right) & if \quad 0 \le \mu \le 1 \\ 4\mu-3 & if \quad \mu \ge 1 \end{cases}$$

This inequality is very much helpful in determining estimates of higher coefficients for some subclasses S.

Now we define some subclasses of S. Let

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in A$$

and satisfy the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in U$$

is univalent starlike function and denoted by

S^* and a subclass $S*(A,B) = \left\{ f(z) \in A, \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right.$ where $-1 \le B < A \le 1, z \in U$

It is obvious that $S^*(A,B)$ is a subclass of S^*.

We introduce a new class as

$$S^*(p) = \left\{ f(z) \in A, \frac{zf'(pz)}{pf(z)} \prec \frac{1+z}{1-z}, z \in U \right\}$$

Symbol $\prec$ stands for subordination.

Analytic bounded functions: Class of analytic bounded function is of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, w(0) = 0, |w(z)| \le 1.$$

It is known that $|c_1| \le 1, |c_2| \le 1 - |c_1|^2$.

2 FEKETE-SZEGO PROBLEM

Our main result is the following

2.1 Theorem

Let the bounded function $w(z) = c_1 z + c_2 z^2 + \cdots$ and $f(z) \in S^*(A,B)(p)$, then

$$\left|a_2-\mu a_3\right|\le\begin{cases}\frac{(A-B)}{(2p-1)}\left[\frac{(A-2Bp)}{(3p^2-1)}-\frac{(A-B)}{(2p-1)}\mu\right] \\ \qquad\qquad if\ \ \mu\le\lambda_1; \\ \frac{A-B}{3p^2-1} \\ \qquad\qquad if\ \ \lambda_1<\mu<\lambda h_2 \\ \frac{(A-B)}{(2p-1)}\left[-\frac{(A-2Bp)}{(3p^2-1)}+\frac{(A-B)}{(2p-1)}\mu\right] \\ \qquad\qquad if\ \ \mu\ge\lambda_2\end{cases}$$

where $\lambda_1=\frac{(A+1)-2p(B+1)(2p-1)}{(A-B)(3p^2-1)}$ and $\lambda_2=\frac{((A-1)+2p(1-B))(2p-1)}{(A-B)(3p^2-1)}$

The results are sharp.

Proof: By definition of $S^*(p)$, we have

$$\frac{zf'(pz)}{pf(z)}\prec\frac{1+Aw(z)}{1+Bw(z)} \qquad \ldots(2)$$

By expending the series (2)

$$1+(2p-1)a_2z+\left((1-2p)a_2^2+(3p^2-1)a_3\right)z^2+\cdots$$
$$=1+(A-B)c_1z+\left((A-B)c_2+B^2c_1^2\right)z^2+\cdots$$ Comparing coefficients of (3)

$$\ldots(3)$$

$$a_2=\frac{(A-B)c_1}{2p-1}\text{ and }a_3=\frac{(A-B)}{(3p^2-1)}c_2+\frac{(B-A)2Bp-AB+A^2}{(3p^2-1)(2p-1)}c_1^2 \qquad \ldots(4)$$

$$\left|a_3-\mu a_2^2\right|\le\frac{(A-B)}{(3p^2-1)}\left|c_2\right|+$$
$$\left\{\frac{(B-A)2Bp-AB+A^2}{(3p^2-1)(2p-1)}-\frac{(A-B)^2}{(2p-1)^2}\mu\right\}\left|c_1^2\right|$$
$$=\frac{(A-B)}{3p^2-1}+$$
$$\left\{\left|\frac{(B-A)2Bp-AB+A^2}{(3p^2-1)(2p-1)}-\frac{(A-B)^2}{(2p-1)^2}\mu\right|-\frac{(A-B)}{3p^2-1}\right\}\left|c_1\right|^2$$
$$\ldots(5)$$

Case 1: when $\mu \le \frac{(2A(1-p)-B)(2p-1)}{(A-B)(3p^2-1)}$

Inequality (5) can be rewritten as

$$\left|a_3-\mu a_2^2\right| \le \frac{(A-B)}{3p^2-1}+ \left\{\frac{(B-A)2Bp-AB+A^2}{(3p^2-1)(2p-1)}-\frac{(A-B)^2}{(2p-1)^2}\mu-\frac{(A-B)}{3p^2-1}\right\} |c_1|^2 \qquad \ldots(6)$$

Sub case 1(a): When $\mu \le \frac{(A+1)-2p(B+1)(2p-1)}{(A-B)(3p^2-1)}$,

Then equation (6) can be rewritten as $\left|a_3-\mu a_2^2\right| \le \frac{(A-B)}{(2p-1)}\left[\frac{(A-2Bp)}{(3p^2-1)}-\frac{(A-B)}{(2p-1)}\mu\right]$ $\ldots(7)$

Sub case 1(b): When $\frac{(A+1)-2p(B+1)(2p-1)}{(A-B)(3p^2-1)} < \mu < \frac{(2A(1-p)-B)(2p-1)}{(A-B)(3p^2-1)}$

then the equation (6) becomes

$$\left|a_3-\mu a_2^2\right| \le \frac{A-B}{3p^2-1} \qquad \ldots(8)$$

Case 2: When $\mu \ge \frac{(2A(1-p)-B)(2p-1)}{(A-B)(3p^2-1)}$,

$$\left|a_3-\mu a_2^2\right| \le \frac{(A-B)}{3p^2-1}+ \left\{-\frac{(B-A)2Bp-AB+A^2}{(3p^2-1)(2p-1)}+\frac{(A-B)^2}{(2p-1)^2}\mu-\frac{(A-B)}{3p^2-1}\right\}|c_1|^2 \qquad \ldots(9)$$

Sub case 2(a): When $\mu \geq \frac{((A-1)+2p(1-B))(2p-1)}{(A-B)(3p^2-1)}$

Then the equation (9) becomes

$$\left|a_3-\mu a_2^2\right| \leq \frac{(A-B)}{(2p-1)}\left[-\frac{(A-2Bp)}{(3p^2-1)}+\frac{(A-B)}{(2p-1)}\mu\right] \quad \ldots(10)$$

Sub case 2(b): When $\frac{(2A(1-p)-B)(2p-1)}{(A-B)(3p^2-1)} < \mu < \frac{((A-1)+2p(1-B))(2p-1)}{(A-B)(3p^2-1)}$,

Then the equation (9) becomes

$$\left|a_3-\mu a_2^2\right| \leq \frac{A-B}{3p^2-1} \quad \ldots(11)$$

Combining the equations (7), (8), (10) and (11).

We get the Fekete Szego inequality for $S^*(A,B)(p)$ as

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{(A-B)}{(2p-1)}\left[\frac{(A-2Bp)}{(3p^2-1)}-\frac{(A-B)}{(2p-1)}\mu\right] \\ \qquad if\ \mu \leq \lambda_1; \\ \frac{A-B}{3p^2-1} \\ \qquad if\ \lambda_1 < \mu < \lambda h_2 \\ \frac{(A-B)}{(2p-1)}\left[-\frac{(A-2Bp)}{(3p^2-1)}+\frac{(A-B)}{(2p-1)}\mu\right] \\ \qquad if\ \mu \geq \lambda_2 \end{cases}$$

(CHANGE)The extremal function for first and third inequality is

$$f_1(z)=z\left\{1+\frac{2p^2z}{(1-3p^2)(2p-1)}\right\}^{\frac{1-3p^2}{p^2}}$$

And extremal function for second inequality is

$$f_2(z) = z\left\{1+2z^2\right\}^{\frac{1}{3p^2-1}}$$

Corollary 1: Putting $A = 1, B = -1$ in the theorem 2.1 we get

$$\left|a_2^2 - \mu a_3\right| \le \begin{cases} \frac{2(2p+1)}{(3p^2-1)(2p-1)} - \frac{4}{(2p-1)^2}\mu & if \quad \mu \le \frac{2p-1}{3p^2-1}; \\ \frac{2}{3p^2-1} & if \quad \frac{2p-1}{3p^2-1} \le \mu \le \frac{2p(2p-1)}{3p^2-1} \\ \frac{4}{(2p-1)^2} - \frac{2(2p+1)}{(3p^2-1)(2p-1)} & if \quad \mu \ge \frac{2p(2p-1)}{3p^2-1} \end{cases}$$

Which is the result obtained by [12].

Corollary 2: Putting $p = 1$ and $A = 1, B = -1$ in the theorem 2.1 we get

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 3 - 4\mu & if \quad \mu \le \frac{1}{2}; \\ 1 & if \quad \frac{1}{2} < \mu < 1; \\ 4\mu - 3 & if \quad \mu \ge 1 \end{cases}$$

Which is the result obtained by [12].

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